EXTENSIONS OF POSITIVE DEFINITE FUNCTIONS ON AMENABLE GROUPS

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Abstract. Let S be a subset of a amenable group G such that $e \in S$ and $S^{-1} = S$. The main result of the paper states that if the Cayley graph of G with respect to S has a certain combinatorial property, then every positive definite operator-valued function on S can be extended to a positive definite function on G. Several known extension results are obtained as a corollary. New applications are also presented.

1. Introduction

Let G be a group. A function $\Phi : G \to \mathcal{L}(\mathcal{H})$ is called *positive definite* if for every $g_1, ..., g_n \in G$ the operator matrix $\{\Phi(g_i^{-1}g_j\}_{i,j=1}^n$ is positive semidefinite. Let $S \subset G$ be a *symmetric* set; that is, $e \in S$ and $S^{-1} = S$. A function $\phi : S \to \mathcal{L}(\mathcal{H})$ is called *(partially) positive definite* is for every $g_1, ..., g_n \in G$ such that $g_i^{-1}g_j \in S$ for all $i, j = 1, ..., n$, $\{\phi(g_i^{-1}g_j\}_{i,j=1}^n$ is a positive semidefinite operator matrix. Extensions of positive definite functions on groups have a long history, starting with the Trigonometric Moment Problem of Carathéodory and Fejér and Krein's Extension Theorem. Recently, it has been proved in [1] that every positive definite operator-valued function on a symmetric interval in an ordered abelian group can be extended to a positive definite function on the whole group. By different techniques, the same extension property was shown to be true in [3] for functions defined on words of length $\leq m$ in the free group with n generators. In the present paper we extend the result to a class of subsets of amenable groups which satisfy a certain combinatorial condition. The result turns out to be more general than the main result in [1] and it is obtained by much simpler means. Our main result was also influenced by [5], where a version of Nehari's Problem was solved for operator functions on totally ordered amenable groups.

2000 *Mathematics Subject Classification.* Primary 43A35; Secondary 47A57, 20E05.

This work has been partially supported by NSF Grant 12-21-11220-N65 and by Grant 2-Cex06-11-34/2006 of the Romanian Government. A visit of the first author to Bucharest has been supported by SOFTWIN.

Let G be a locally compact group. A *right invariant mean* m on G is a state on $L^\infty(G)$ which satisfies

$$m(f) = m(f_x),$$

for all $x \in G$, where $f_x(y) = f(yx)$. In case there exists a right invariant mean on G, G is called *amenable*. We will occasionally write $m^x(f(x))$ for $m(f)$. There exist many other equivalent characterizations of amenability [4].

For graph theoretical notions we refer the reader to [6]. By a *graph* we mean a pair $G = (V, E)$ in which V is a set called the *vertex set* and E is a symmetric nonreflexive binary relation on V, called the *edge set*. We consider in general the vertex set to be infinite. A graph is called *chordal* if every finite simple cycle $[v_1, v_2, ..., v_n, v_1]$ in E with $n \geq 4$ contains a chord, i.e. an edge connecting two nonconsecutive vertices of the cycle. Chordal graphs play an important role in the extension theory of positive definite matrices ([7] and [9]).

Let G be a group. If $S \subset G$ is symmetric, we define the *Cayley graph* of G with respect to S (denoted $\Gamma(G, S)$) as the graph whose vertices are the elements of G, while $\{x, y\}$ is an edge iff $x^{-1}y \in S$.

2. The main result

The basic result of the paper is the following.

Theorem 2.1. *Suppose G is amenable, and $S \subset G$. If $\Gamma(G, S)$ is chordal, then any positive definite function ϕ on S admits a positive definite extension Φ on G.*

Proof. Consider the partially positive semidefinite kernel $k : G \times G \to \mathcal{L}(\mathcal{H})$, defined only for pairs (x, y) for which $x^{-1}y \in S$, by the formula

$$k(x, y) = \phi(x^{-1}y).$$

Since the pattern of specified values for this kernel is chordal by assumption, it follows from [9] that k can be extended to a positive semidefinite kernel $K : G \times G \to \mathcal{L}(\mathcal{H})$. Note that $K(x, y)$ has no reason to depend only on $x^{-1}y$.

For any $x, y \in G$, the operator matrix $\begin{pmatrix} \phi(e) & K(x,y) \\ K(x,y)^* & \phi(e) \end{pmatrix}$ is positive semidefinite, whence it follows that $K(x, y)^* K(x, y) \leq \phi(e)^2$. In particular, all operators $K(x, y)$, $x, y \in G$, are bounded by a common constant.

Fix then $\xi, \eta \in \mathcal{H}$ and $x \in G$. The function $F_{x;\xi,\eta} : G \to \mathbb{C}$, defined by

$$F_{x;\xi,\eta}(y) = \langle K(yx, y)\xi, \eta\rangle \tag{2.1}$$

is in $L^\infty(G)$. Define then $\Phi : G \to \mathcal{L}(\mathcal{H})$ by

$$\langle \Phi(x)\xi, \eta\rangle = m(F_{x;\xi,\eta}). \tag{2.2}$$

We claim that Φ is a positive definite function. Indeed, take arbitrary vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$. We have

$$\sum_{i,j=1}^{n} \langle \Phi(g_i^{-1} g_j)\xi_i, \xi_j\rangle = \sum_{i,j=1}^{n} m(F_{g_i^{-1}g_j;\xi_i,\xi_j}) = \sum_{i,j=1}^{n} m^y \left(\langle K(yg_i^{-1}g_j, y)\xi_i, \xi_j\rangle\right).$$

Take one of the terms in the last sum; the mean m is applied to the function $y \mapsto \langle K(yg_i^{-1}g_j, y)\xi_i, \xi_j\rangle$. The right invariance of m implies that we may apply the change of variable $z = yg_i^{-1},\ y = zg_i$, and thus

$$m^y \left(\langle K(yg_i^{-1}g_j, y)\xi_i, \xi_j\rangle\right) = m^z \left(\langle K(zg_j, g_i z)\xi_i, \xi_j\rangle\right).$$

Therefore

$$\sum_{i,j=1}^{n} \langle \Phi(g_i^{-1} g_j)\xi_i, \xi_j\rangle = \sum_{i,j=1}^{n} m\Big(\langle K(zg_j, g_i z)\xi_i, \xi_j\rangle\Big) = m\Big(\sum_{i,j=1}^{n} \langle K(zg_j, g_i z)\xi_i, \xi_j\rangle\Big).$$

But the positivity of K implies that, for each $z \in G$,

$$\sum_{i,j=1}^{n} \langle K(zg_j, g_i z)\xi_i, \xi_j\rangle \geq 0.$$

Since m is a positive functional, it follows that indeed Φ is positive definite. On the other hand, for $x \in S$, the function $F_{x;\xi,\eta}$ is constant, equal to $\langle \phi(x)\xi, \eta\rangle$. Therefore Φ is indeed the desired extension of ϕ. ∎

Remark 2.2. The chordality of $\Gamma(G,S)$ means that for every finite cycle $[g_1, ..., g_n, g_1]$, $n \geq 4$, at least one $\{g_i, g_{i+2}\}$ (with $g_{n+1} = g_1$ and $g_{n+2} = g_2$) is an edge. Denoting $\xi_k = g_k g_{k+1}^{-1}$, the condition is equivalent to: $\xi_1, ..., , \xi_n \in S$, $\xi_1\xi_2 \cdots \xi_n = e$, $n \geq 4$, implies that there exist $i = 1, ..., m$ such that $\xi_i\xi_{i+1} \in S$ (here $\xi_{n+1} = \xi_1$).

Remark 2.3. Let $\Lambda \subset G$ be such that $e \in \Lambda$, and e cannot be written as a product of elements in Λ different from e, and let $S = \Lambda\Lambda^{-1}$. Assume we have that $S = \Lambda \cup \Lambda^{-1}$. Then $\xi_1\xi_2 \cdots \xi_n = e$, with $\xi_1, ..., \xi_n \in S$, implies the existence of k such that $\xi_k \in \Lambda$ and $\xi_{k+1} \in \Lambda^{-1}$, thus $\xi_k\xi_{k+1} \in S$, implying $\Gamma(G,S)$ is chordal.

We conjecture the following reciprocal of Theorem 2.1.

Conjecture 2.4. *For every $S \subset G$ such that $\Gamma(G,S)$ is not chordal there exists a positive definite function $\phi : S \to \mathcal{L}(\mathcal{H})$ which does admit a positive definite extension to G.*

The following examples strongly suggest that the above conjecture has a positive answer. Let $G = \mathbb{Z}^2$ and let $S = \mathbb{Z}^2 - \{(1,1), (-1,-1)\}$, the minimal number of points that can be excluded. Then $(0,0)$, $(0,1)$, $1,1)$, and $(-1,0)$ form a chordless cycle of length 4 in $\Gamma(G,S)$. Define $\phi : S \to M_2(\mathbb{C})$ by $\phi((0,0)) =$

$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\phi((1,0)) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, $\phi((0,1)) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, and $\phi(g') = 0$, for every $g' \in S - \{(0,0),(1,0),(-1,0),(0,1),(0,-1)\}$. Let K be a maximal clique of $\Gamma(G,S)$. We may assume that $(0,0) \in K$, in which case $(1,1) \notin K$. This fact implies that the matrix $\{\phi(x-y)\}_{x,y\in K}$ can be written as a direct sum of copies of $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, so ϕ is positive definite. Assume that ϕ admits a positive definite extension Φ to G. Then, since

$$\begin{pmatrix}\Phi((0,0)) & \Phi((1,0))^* & \Phi((1,1))^* \\ \Phi((1,0)) & \Phi((0,0)) & \Phi((0,1))^* \\ \Phi((1,1)) & \Phi((0,1)) & \Phi((0,0))\end{pmatrix} \geq 0$$

and

$$\begin{pmatrix}\Phi((0,0)) & \Phi((0,1))^* & \Phi((1,1))^* \\ \Phi((0,1)) & \Phi((0,0)) & \Phi((1,0))^* \\ \Phi((1,1)) & \Phi((1,0)) & \Phi((0,0))\end{pmatrix} \geq 0,$$

it follows that $\Phi((1,1)) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Since

$$\begin{pmatrix}\Phi((0,0)) & \Phi((1,1))^* & \Phi((2,1)))^* \\ \Phi((1,1)) & \Phi((0,0)) & \Phi((1,1))^* \\ \Phi((2,1)) & \Phi((1,1)) & \Phi((0,0))\end{pmatrix} \geq 0$$

the $(2,1)$ entry of $\Phi((2,1))$ equals 1, contradicting the fact that $\Phi((2,1)) = \phi((2,1)) = 0$. This implies that ϕ does not admit a positive definite extension to $\mathbb{Z}^2$.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite set. By the definition introduced in [8], a sequence $\{c_k\}_{k\in\Lambda-\Lambda}$ of complex numbers is called *positive definite with respect to* Λ if the matrix $\{c_{k-l}\}_{k,l\in\Lambda}$ is positive definite. This definition is weaker then the one used in this paper, since it requires only a single matrix built on the given data to be positive definite. A finite subset $\Lambda \subset \mathbb{Z}^d$ is said to posses the *extension property* if every sequence $\{c_k\}_{k\in\Lambda-\Lambda}$ admits a positive extension to $\mathbb{Z}^d$. Let $R(0,n) = \{0\} \times \{0,1,...,n\}$, $R(1,n) = \{0,1\} \times \{0,1,...,n\}$, and $S(1,n) = R(1,n) - \{(1,n)\}$. The following is the main result of [2].

Theorem 2.5. *A finite $\Lambda \subset \mathbb{Z}^2$ has the extension property if and only if Λ is the translation by a vector in $\mathbb{Z}^2$ of a set isomorphic to one of the following sets: $R(0,n)$, $R(1,n)$, or $S(1,n)$, $n \geq 0$.*

Let $\Lambda = R(1,n)$, when $S = \Lambda - \Lambda = \{-1,0,1\} \times \{-n,...,0,...,n\}$. By the previous theorem, every scalar positive definite sequence with respect to Λ on S admits a positive definite extension to $\mathbb{Z}^2$. The points $(0,0)$, $(-1,n)$, $(0,2n)$, and $(1,n)$ form a chordless cycle in $\Gamma(\mathbb{Z}^2,S)$, and for every Hilbert space $\mathcal{H}$ with $\dim\mathcal{H} \geq 2$, there exists a sequence $\{C_k\}_{k\in S}$ of operators on $\mathcal{H}$ which is positive definite (in the

stronger sense), but does not admit a positive definite extension to $\mathbb{Z}^2$. The same is true for the sets $S(1,n)$ as well. We will present next the details concerning the different behaviour of scalar and operator sequences for a subset of $\mathbb{Z}^2$ not covered by Theorem 2.5.

Let $G=\mathbb{Z}^2$, $m,n\in\mathbb{N}$, $m,n\geq 2$, and let S consist of the points $(k,0)$, $|k|\leq m$ together with the points $(0,l)$, $|l|\leq n$. Let $\{C_{kl}\}_{(k,l)\in S}$ be a positive definite sequence of operators. The positive definiteness condition is equivalent to

$$\begin{pmatrix} C_{00} & C_{10}^* & \cdots & C_{m0}^* \\ C_{10} & C_{00} & \cdots & C_{m-1,0}^* \\ \vdots & \ddots & \ddots & \vdots \\ C_{m0} & C_{m-1,0} & \cdots & C_{00} \end{pmatrix} \geq 0 \tag{2.3}$$

and

$$\begin{pmatrix} C_{00} & C_{01}^* & \cdots & C_{0n}^* \\ C_{01} & C_{00} & \cdots & C_{0,n-1}^* \\ \vdots & \ddots & \ddots & \vdots \\ C_{0n} & C_{0,n-1} & \cdots & C_{00} \end{pmatrix} \geq 0. \tag{2.4}$$

In case $\{c_{kl}\}_{(k,l)\in S}$ is the sequence defined by $c_{k0}=e^{ik\alpha}$ and $c_{0l}=e^{il\beta}$, the matrices in (2.3) are rank 1 positive definite Toeplitz matrices and $c_{kl}=e^{ik\alpha}e^{il\beta}$, $(k,l)\in\mathbb{Z}^2$ is a positive definite extension to $\mathbb{Z}^2$ of the initial sequence. It is a classical result of Carathéodory and Fejér that every positive definite Toeplitz matrix is a positive linear combination of rank 1 positive definite Toeplitz matrices. This implies that the positive semidefiniteness of the matrices in (2.3) guarantees the existence of a positive definite extension to $\mathbb{Z}^2$ of every positive definite sequence $\{c_{kl}\}_{(k,l)\in S}$ of complex numbers.

Let U_1 and U_2 be two noncommuting unitary operators on a Hilbert space $\mathcal{H}$ with $\dim\mathcal{H}\geq 2$. Defining $C_{00}=I$, $C_{k0}=U_1^k$, and $C_{0l}=U_2^l$, the matrices in (2.3) and (2.4) are positive semidefinite. Assuming the sequence $\{C_{kl}\}_{(k,l)\in S}$ admits a positive definite extension to $\mathbb{Z}^2$, the operator C_{11} has to simultaneously verify the conditions $\begin{pmatrix} C_{00} & C_{01}^* & C_{11}^* \\ C_{01} & C_{00} & C_{10}^* \\ C_{11} & C_{10} & C_{00} \end{pmatrix}\geq 0$ and $\begin{pmatrix} C_{00} & C_{10}^* & C_{11}^* \\ C_{10} & C_{00} & C_{01}^* \\ C_{11} & C_{01} & C_{00} \end{pmatrix}\geq 0$. For our data, the above conditions are equivalent to $C_{11}=U_2U_1$, respectively $C_{11}=U_2U_1$, which is false, since U_1 and U_2 do not commute. Thus $\{C_{kl}\}_{(k,l)\in S}$ does not admit any positive definite extension to $\mathbb{Z}^2$.

Proposition 2.6. *Let $0 \in S = -S$ be a finite subset of $\mathbb{Z}^2$ such that $\Gamma(\mathbb{Z}^2, S)$ is chordal and S spans $\mathbb{Z}^2$. Then S is infinite.*

Proof. Suppose $S \subset \mathbb{Z}^2$ is finite and $\Gamma(\mathbb{Z}^2, S)$ is chordal. There are a finite number of directions among the elements of S; suppose the elements of maximum length in each of these directions, together with their inverses, are enumerated $s_1, s_2, \dots, s_{2n}$ in the order of their arguments.

For a positive integer N consider the cycle $[x_0, x_2, \dots x_{2nN-1}, x_0]$ in $\Gamma(\mathbb{Z}^2, S)$, defined as follows: $x_0 = 0$, $x_k - x_{k-1} = s_j$ if $(j-1)N < k \le jN$. We claim that, if N sufficiently large, this is a cycle with no chords.

Indeed, suppose $\{x_k, x_l\}$ is an edge with $l - k \ge 2$. The points $x_0, \dots, x_{2nN-1}$ form a polygon P with $2n$ sides A_j parallel to s_j respectively, each side containing N points x_k. We have the following possibilities:

—If x_k and x_l are on the same side A_j of P, then $x_l - x_k = (l-k)s_j$ would be an element of S colinear with s_j, but longer, which is not possible.

—If $x_k \in A_j$, $x_l \in A_{j+1}$, then the argument of $x_l - x_k$ would be strictly between the arguments of s_j and s_{j+1}: again a contradiction.

—Finally, we may chose N sufficiently large such that, if x_k and x_l are on nonconsecutive sides of P, then $x_l - x_k$ has length larger than any element of S.

So the cycle obtained has no chords, contrary to the chordality assumption in the hypothesis. Thus S must be infinite. ∎

Remark 2.7. If Conjecture 2.4 is true, then Lemma 2.6 would imply that for every finite $S \subset \mathbb{Z}^2$ such that $0 \in S = -S$ and S spans $\mathbb{Z}^2$, there exists a positive definite function on S which does not admit a positive definite extension to $\mathbb{Z}^2$.

3. Applications

3.1. **Ordered groups and related questions.** Suppose G is a (left or right) totally ordered group. Take $a \in G$, $a \ge e$, and define $\Lambda = [e, a)$, and $S = (a^{-1}, a)$. Then e cannot be written as a product of elements in Λ and $S = \Lambda\Lambda^{-1} = \Lambda \cup \Lambda^{-1}$. Then by Remark 2.3 the graph $\Gamma(G, S)$ is chordal. Thus, in an amenable totally ordered group any positive definite function defined on a symmetric interval can be extended to the whole group.

The same argument yields the following more general result.

Proposition 3.1. *Suppose G is amenable, while G' is a totally ordered group, with unit e'. Let $g : G \to G'$ be a group morphism. Take $a' \in G'$, $a' \ge e'$, and $S = g^{-1}((a'^{-1}, a'))$. Then any positive definite operator function on S can be extended to a positive definite function on the whole group.*

The above proposition has the following consequence which represents the main result of [1]. The proof derived here is much simpler.

Corollary 3.2. *Let G_1 be a totally ordered abelian group, $a \in G_1$, $a > 0$, and let G_2 be an abelian group. Then any positive definite operator function on $(-a, a) \times G_2$ can be extended to a positive definite function on $G_1 \times G_2$.*

Several well-known results, such that the Classical Trigonometric Moment Problem and Krein's Extension Theorem are particular cases of Corollary 3.2. Another simple application of Corollary 3.2 is the following. Take $\alpha, \beta \in \mathbb{R}$, and define $g : \mathbb{Z}^2 \to \mathbb{R}$ by $g(m, n) = \alpha m + \beta n$. Thus, all positive definite functions defined on the strip $|\alpha m + \beta n| < a$ can be extended to a positive definite function on $\mathbb{Z}^2$.

A more interesting example for Proposition 3.1 is given by the Heisenberg group H over the integers. This is the group of matrices of the form

$$X_{m,n,p} = \{\left(\begin{smallmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{smallmatrix}\right),\ m, n, p \in \mathbb{Z}\}.$$

It is an amenable group, and for any $\alpha, \beta \in \mathbb{R}$, we can consider the morphism $g : H \to \mathbb{R}$, given by $g(X_{m,n,p}) = \alpha m + \beta n$. Thus any positive definite function defined on the set $\{X_{m,n,p} : |\alpha m + \beta n| < a\}$ can be extended to a positive definite function on H.

3.2. **Trees and Cayley graphs.** For this application we need some supplementary preliminaries. If $\Gamma = (V, E)$ is a graph, the distance $d(v, w)$ between two vertices is defined as

$$d(v, w) = \min\{n : \exists v = v_0, v_1, \dots, v_n = w,\ \text{such that } \{v_i, v_{i+1}\} \in E(\Gamma)\}.$$

We define the graph $\hat{\Gamma}_n$ that has the same vertices as Γ, while $\{v, w\}$ is an edge of $\hat{\Gamma}_n$ if and only if $d(v, w) \leq n$.

A graph without any simple cycle is called a *tree*. If x and y are two distinct vertices of a tree, then $P(x, y)$ denotes the unique simple path joining x and y.

Lemma 3.3. *If Γ is a tree, then $\hat{\Gamma}_n$ is chordal for any $n \geq 1$.*

Proof. Take a minimal cycle C of length > 3 in $\hat{\Gamma}_n$. Suppose $x, y \in C$ maximize the distance between any two points of C. If $d(x, y) \leq n$, then C is a clique, which is a contradiction. Thus x and y are not adjacent in $\hat{\Gamma}_n$. Suppose v, w are the two vertices of $\hat{\Gamma}_n$ adjacent to x in the cycle C. Now $P(x, v)$ has to pass through a vertex which is on $P(x, y)$, since otherwise the union of these two paths would be the minimal path connecting y and v, and it would have length strictly larger than $d(x, y)$. Denote by v_0 the element of $P(x, v) \cap P(x, y)$ which has the largest

distance to x; since $d(y,v) = d(y,v_0) + d(v_0,v) \leq d(y,x) = d(y,v_0) + d(v_0,x)$, it follows that $d(v_0,v) \leq d(v_0,x)$.

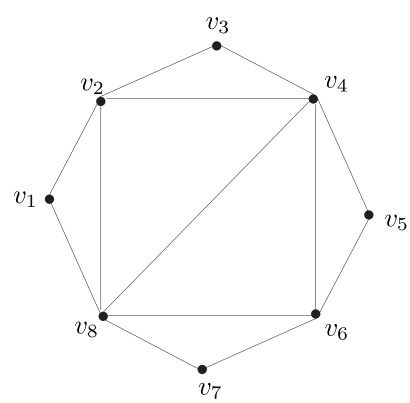
Similarly, if w_0 is the element of $P(x,w) \cap P(x,y)$ which has the largest distance to x, it follows that $d(w_0,w) \leq d(w_0,x)$.

Suppose now that $d(v_0,x) \leq d(w_0,x)$. Then

$$\begin{aligned} d(v,w) &= d(v,v_0) + d(v_0,w_0) + d(w_0,w) \\ &\leq d(x,v_0) + d(v_0,w_0) + d(w_0,w) = d(x,w) \leq n, \end{aligned}$$

since w is adjacent to x. Then $(v,w) \in E$, and C is not minimal: a contradiction. Thus $\hat{\Gamma}_n$ is chordal. ∎

It is worth mentioning that Γ chordal does not necessarily imply $\hat{\Gamma}_n$ chordal. For instance, the graph Γ below is chordal, but $\hat{\Gamma}_2$ is not, since it has $[v_1, v_3, v_5, v_7]$ as a 4-minimal cycle.



Suppose now that the group G is finitely generated by a set A with $A = A^{-1}$. The length of an element $x \in G$ is defined by

$$l(x) = \min\{n : x = b_1 \cdots b_n,\ b_i \in A\};$$

it is equal to the distance between x and e in the Cayley graph $\Gamma(G,A)$. If $\Gamma(G,A)$ is a tree, then Lemma 3.3 and Theorem 2.1 yield the following result.

Proposition 3.4. *Suppose G is amenable and $\Gamma(G,A)$ is a tree. If $S = \{x \in \Gamma : l(x) \leq n\}$, then any positive definite function on S can be extended to the whole of G.*

The proposition applies to the free product $G = \mathbb{Z}_2 \star \mathbb{Z}_2$: it is easily seen that, if A is formed by the two generators, then $\Gamma(G,A)$ is order isomorphic to $\mathbb{Z}$, and is thus a tree. So any positive definite function defined on words of length smaller or equal to n extends to the whole group.

Unfortunately, there seem not to be many amenable graphs whose Cayley graph with respect to some set of generators is a tree. Note first the following simple lemma.

Lemma 3.5. *Suppose G is a group, $A \subset G$ is a set of generators, and $\Gamma(G,A)$ is a tree.*

(i) For every $x \in G$, there is a unique way of writing $x = a_1 \cdots a_n$, with $a_i \in A$, and $a_i a_{i+1} \neq e$; moreover, $l(x) = n$. (We call $a_1, a_2, ..., a_n$ the letters of x.)

(ii) Take $x \in G$, with a_x the first letter of x. If $y \in G$, and the last letter of y is not a_x^{-1}, then $l(yx) = l(x) + l(y)$.

We can then obtain the following proposition.

Proposition 3.6. *Suppose that G is a discrete amenable group, and $A \subset G$ is a subset of generators, such that $\Gamma(G,A)$ is a tree. Then either $G = \mathbb{Z}$, or $G = \mathbb{Z}_2 \star \mathbb{Z}_2$.*

Proof. Note first that G cannot be finite, since then we may take an element $a \in A$ with finite order p, and construct the cycle $[e, a, a^2, \dots, a^{p-1}]$ in $\Gamma(G,A)$, which has no chords.

One of the alternate definitions of an amenable group is the Følner condition, which in the case of discrete groups can be stated as follows: given any finite set $F \subset G$ and any $\epsilon > 0$, there exists a finite subset $K \subset G$, such that

$$\frac{\mathrm{card}(K \triangle FK)}{\mathrm{card} K} < \epsilon$$

($K \triangle FK$ is the symmetric difference). Using a translation, if necessary, we may assume $e \in K$. Denote also $S_n = \{x \in G : l(x) = n\}$.

Suppose that $x \in G$; Lemma 3.5 implies that there is at most one element $a \in A$ with the property that $l(ax) \neq l(x) + 1$ (otherwise there would exist a cycle in $\Gamma(G,A)$). Therefore, if $x \in S_n$, there is at most one $a \in A$ such that $ax \notin S_{n+1}$. Moreover, if $x, y \in S_n$, $x \neq y$, $a, b \in A$ with $ax, by \in S_{n+1}$, then $ax \neq by$ (otherwise we obtain again a cycle in $\Gamma(G,A)$.

It follows then that, if A has at least 3 elements, then, for any finite set $E \subset S_n$, $AE \cap S_{n+1}$ has at least twice more elements than E. Therefore

$$\mathrm{card} K = \sum_n \mathrm{card}(K \cap S_n) \leq 2 \sum_n \mathrm{card}(AK \cap S_{n+1}) \leq 2\mathrm{card}(AK). \tag{3.1}$$

Thus $\mathrm{card}(K \triangle AK) \geq \mathrm{card} K$, and the Følner condition cannot be satisfied.

Therefore A has at most two elements. If it has only one element, then, being infinite, it is $\mathbb{Z}$.

Suppose it has two elements. If $a^2 \neq e$ and $x \in G$, then applying again Lemma 3.5, we have that $l(a'x) \neq l(x) + 2$ for at most one element a' in the

set $A' = \{a^2, ab, ba\}$, and for $x, y \in S_n$, $x \neq y$, $a', b' \in A'$ with $a'x, b'y \in S_{n+2}$, we have $a'x \neq b'y$. Therefore, for any finite set $E \subset S_n$, $AE \cap S_{n+2}$ has at least twice more elements than E, and we obtain (3.1) with S_{n+1} replaced by S_{n+2}. Thus again $\mathrm{card}(K \triangle AK) \geq \mathrm{card} K$, and the Følner condition cannot be satisfied.

Since a similar argument applies in case $b^2 \neq e$, the only remaining possibility is $a^2 = b^2 = e$. Now if either ab or ba would have finite order, this would produce a cycle in $\Gamma(G, A)$. Thus they are both of infinite order, and it follows easily that G is isomorphic to $\mathbb{Z}_2 \star \mathbb{Z}_2$. ∎

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